Generalisation of the Painleve test

1986 J. Phys. A: Math. Gen. 19 L381

(http://iopscience.iop.org/0305-4470/19/7/003)

J. Phys. A: Math. Gen. 19 (1986) L381-L384. Printed in Great Britain

LETTER TO THE EDITOR

Generalisation of the Painlevé test

Sanjay Puri

Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, 1110 W Green Street, Urbana, Illinois 61801, USA

Received 13 December 1985

Abstract. We suggest a generalisation of the Painlevé test to include situations in which the leading order singularity is not determined by a simple singularity analysis. Such situations often arise in equations whose non-linearities are convective derivatives (e.g. Hasegawa-Mima equation in plasma physics).

As yet, there is no systematic method for determining whether or not a dynamical system is integrable. However, recent work (e.g. Bountis 1984, Chang *et al* 1982) indicates that there is a connection between integrability of a dynamical system and the analytic structure of its equations of motion. The Painlevé test was originally proposed in connection with partial differential equations solvable by the inverse scattering transform. It was conjectured that if all possible reductions (perhaps after a change of variables) of a partial differential equation to an ordinary differential equation had the Painlevé property (i.e. the only movable singularities of the solution in the complex time plane were simple poles (Ince 1956)), then the original partial differential equation was solvable by the inverse scattering transform. A generalised version of this test, which is directly applicable to partial differential equations, was proposed by Weiss *et al* (1983). A crucial element in the application of this test is the determination of the leading order singularity (usually specified by the exponent p) of the series solution. However, there are equations in which the leading order singular terms (resulting from a series expansion) identically cancel. This is often the case in equations whose non-linearities are the convective derivative. A classic example is that of the Hasegawa-Mima equation (Hasegawa and Mima 1978), which describes the propagation of 2D drift waves in a plasma. This equation is known to have vortex solutions translating steadily along the x direction (Larichev and Reznik 1976, Makino *et al* 1981b). These show strong stability under collision (Makino *et al* 1981a, b), behaving in a manner reminiscent of 2D solitons. This led Ichikawa *et al* (1983) to speculate that the Hasegawa-Mima equation may be integrable. Unfortunately, we cannot apply the Painlevé test of Weiss *et al* (1983) to equations of the Hasegawa-Mima type. This is because we are unable to determine p in the usual way, viz by demanding that the leading order terms be equally singular. In this letter, we propose a generalisation of the Painlevé test so that it can be applied to such equations. We then illustrate the proposed generalisation by means of a simple model equation.

The proposed generalisation is as follows.

If p is undetermined by the leading order singularity analysis, take p as arbitrary and proceed in the usual way. In general, the expansion coefficients will now satisfy partial differential equations. One of the required arbitrary functions will be comprised of the infinite constants of integration of these differential equations. The others appear at 'resonance' situations. If we determine that all values of p (and there is at least one such) satisfying 'resonance' are integer and make all expansion coefficients single-valued, then the considered system is integrable.

The condition on all values of p satisfying 'resonance' is similar to that of Chang *et al* (1982), who find that more than one singular expansion is possible for the Hénon–Heiles equation.

As an example, we consider a model dispersive equation, with the relevant feature

$$u_t + u_x u_{tt} = u_t u_{xt} + u_{xx}. \qquad (1)$$

Application of the Painlevé test proceeds by attempting to construct a consistent, single-valued solution about a movable singularity manifold (e.g. Weiss *et al* 1983). The Cauchy–Kowalesky theorem would require that we have the correct number of arbitrary functions (two in this case) in the expansion. To facilitate application of the test, we rewrite (1) as

$$\Psi = u_t \qquad (2a)$$

$$\Psi + u_x \Psi_t = \Psi\Psi_x + u_{xx}. \qquad (2b)$$

Expand ψ and u about the arbitrary, yet well-behaved in x and t, singularity manifold ($\theta(x, t) = 0$) as

$$\psi = \sum_{n=0}^{\infty} \psi_n(x, t)\theta(x, t)^{n-p-1} \qquad (3a)$$

$$u = \sum_{n=0}^{\infty} u_n(x, t)\theta(x, t)^{n-p}. \qquad (3b)$$

Here, p measures the order of the singularity. Substitution of (3) into (2) gives

$$\sum_{n=0}^{\infty} [\psi_n\theta^{n-p-1} = (n-p)u_n\theta^{n-p-1}\theta_t + u_{n,t}\theta^{n-p}] \qquad (4a)$$

and

$$\begin{aligned}\sum_{n,m=0}^{\infty} [(n-p)(m-p-1)u_n\psi_m\theta^{n+m-2p-3}\theta_x\theta_t + (n-p)u_n\psi_{m,t}\theta^{n+m-2p-2}\theta_x \\ + (m-p-1)u_{n,x}\psi_m\theta^{n+m-2p-2}\theta_t + u_{n,x}\psi_{m,t}\theta^{n+m-2p-1} \\ = (n-p)(m-p-1)u_n\Psi_m\theta^{n+m-2p-3}\theta_x\theta_t + (n-p)u_n\Psi_{m,x}\theta^{n+m-2p-2}\theta_t \\ + (m-p-1)u_{n,t}\psi_m\theta^{n+m-2p-2}\theta_x + u_{n,t}\psi_{m,x}\theta^{n+m-2p-1}] \\ + \sum_{n=0}^{\infty} [(n-p)(n-p-1)u_n\theta^{n-p-2}\theta_x^2 + 2(n-p)u_{n,x}\theta^{n-p-1}\theta_x \\ + (n-p)u_n\theta^{n-p-1}\theta_{xx} - \Psi_n\theta^{n-p-1} + u_{n,xx}\theta^{n-p}].\end{aligned} \qquad (4b)$$

In the usual case (Weiss *et al* 1983) we expect (4b) to fix the value of p. In this case, it does not. This is because the most singular terms cancel out. However, from (4b) it is clear that p cannot take non-integer values. If p were non-integer, the expansion terms from the linear (u_t and u_{xx}) and non-linear ($u_x u_{xt}$ and $u_t u_{xt}$) terms in (4b) would have to be independently matched. But the expansion for the linear diffusion terms is consistent only for $p=-1$, leading to a contradiction. Furthermore, the only non-positive values p can take are 0 and -1. For these, (1) can easily be shown to have the Painlevé property. Taking p as an arbitrary positive integer, we can find the recursion relations for u_k, ψ_k from (4). Equating coefficients of θ^{k-p-1} in (4a) yields

$$\psi_k=(k-p)u_k\theta_t+u_{k-1,t}. \tag{5a}$$

Equating coefficients of θ^{k-2p-2} in (4b) yields

$$\sum_{n=0}^{k}[(n-p)u_n(\psi_{k-n,t}\theta_x-\psi_{k-n,x}\theta_t)+(k-n-p-1)\Psi_{k-n}(u_{n,x}\theta_t-u_{n,t}\theta_x)]$$
$$+\sum_{n=0}^{k-1}(u_{n,x}\Psi_{k-n-1,t}-u_{n,t}\Psi_{k-n-1,x})=(k-2p)(k-2p-1)u_{k-p}\theta_x^2$$
$$+2(k-2p-1)u_{k-p-1,x}\theta_x+(k-2p-1)u_{k-p-1}\theta_{xx}-\Psi_{k-p-1}+u_{k-p-2,xx}. \tag{5b}$$

For $k=0$, we have from (5)

$$\psi_0=-pu_0\theta_t \tag{6a}$$

$$pu_0(\psi_{0,t}\theta_x-\psi_{0,x}\theta_t)+(p+1)\psi_0(u_{0,x}\theta_t-u_{0,t}\theta_x)=0. \tag{6b}$$

Substituting in (6b) for ψ_0, $\psi_{0,x}$ and $\psi_{0,t}$ from (6a), we have

$$pu_0(\theta_{xt}\theta_t-\theta_x\theta_{tt})=\theta_t(u_{0,x}\theta_t-u_{0,t}\theta_x). \tag{7}$$

In the usual case (Weiss *et al* 1983) u_0 (and higher u_k) are algebraically determined. Here, we have a differential equation for u_0 because of the cancellation of the leading order terms in (4b). Solution of (7) would, in general, involve a constant of integration. If we assume, following Jimbo *et al* (1982), that the conditions of the implicit function theorem hold on the singularity manifold, then we can solve for $x=g(t)$ and consider only $u_k\equiv u_k(t)$. This reduces (7) to

$$\frac{1}{u_0}\frac{\mathrm{d}u_0}{\mathrm{d}t}=\frac{pg_{tt}}{g_t}$$

with the solution

$$u_0=Ag_t^p$$

where A is a constant of integration. As p is integer, u_0 will be single-valued.

If we substitute in (5b) for ψ_k, $\psi_{k,x}$ and $\psi_{k,t}$ from (5a) we get the general differential equation

$$u_{k,x}u_0\theta_t^2p(k+1)-u_{k,t}u_0\theta_x\theta_tp(k+1)+u_ku_0(\theta_{xt}\theta_t-\theta_x\theta_{tt})p(k+1)(k-p)$$
$$=G(u_i,\ \theta \text{ and their derivatives}) \tag{8}$$

where G is some complicated function of its arguments and i goes from 1 to $k-1$. 'Resonances' arise when the left-hand side vanishes. These correspond to stages in the expansion at which arbitrary functions can appear. In this case, we have only one 'resonance' and it is at $k=-1$, corresponding to the arbitrary singularity manifold.

The second arbitrary function appears in the form of the infinite constants of integration arising from the solutions of (8). This was suggested by Friedman (1984).

Under the ansatz of Jimbo *et al* (1982), (8) simplifies to

$$u_{k,t}+(k-p)u_k\frac{g_{tt}}{g_t}=\frac{-G}{(pk+1)u_0g_t}. \tag{9}$$

This equation has the formal solution (Ince 1956)

$$u_k=Kg_t^{p-k}-\frac{g_t^{p-k}}{Ap(k+1)}\int Gg_t^{k-2p-1}\,\mathrm{d}t \tag{10}$$

where K is an arbitrary constant of integration. The form of G ensures that it is a single-valued function of its arguments (viz $u_{1\to k-1}$, θ and their derivatives). We have seen that u_0 is single-valued. Therefore u_1, u_2, etc, are single-valued for all admissible (integer) values of p.

Thus, we have the requisite number of arbitrary functions and are able to perform a consistent single-valued expansion about the singularity manifold. By our extension of the Painlevé test, (1) has the Painlevé property and is integrable.

A similar conclusion is arrived at by considering a typical reduction of (1), obtained by looking for travelling wave solutions $u\equiv u(x-vt)$. Then the non-linear terms in (1) cancel out. The remaining linear equation has the Painlevé property (Ince 1956). Thus, according to the original version of the Painlevé test, our model equation is integrable.

The author gratefully acknowledges numerous helpful discussions with B Friedman, who introduced him to both the Hasegawa–Mima equation and the Painlevé test. He also wishes to thank Y Oono and E A Jackson for useful suggestions.

This work was supported in part by the National Science Foundation under MRL Grant DMR-83-16981.

References

Bountis T 1984 in *Singularities and Dynamical Systems* ed S Pneumatikos (Amsterdam:North-Holland)
Chang Y F, Tabor M and Weiss J 1982 *J. Math. Phys.* **23** 531
Friedman B 1984 Personal communication
Hasegawa A and Mima K 1978 *Phys. Fluids* **21** 87
Ichikawa Y, Konno K and Wadati M 1983 *Long-Time Predictions in Dynamics* (New York: Wiley-Interscience) pp 345–65
Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
Jimbo M, Kruskal M D and Miwa T 1982 *Phys. Lett.* **92A** 59
Larichev V D and Reznik G M 1976 *Dokl. Akad. Nauk.* **231** 1077
Makino M, Kanimura T and Sato T 1981a *J. Phys. Soc. Japan* **50** 954
Makino M, Kanimura T and Taniuti R 1981b *J. Phys. Soc. Japan* **50** 980
Weiss J, Tabor M and Carnevale A 1983 *J. Math. Phys.* **24** 522